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Harness Processes and Non-Homogeneous Crystals

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Abstract We consider the Harmonic crystal, a measure on $\mathbb{R}^{\mathbb{Z}^d}$ with Hamiltonian $H(\mathbf{x}) = \sum_{i,j} J_{i,j}(\mathbf{x}(i) - \mathbf{x}(j))^2 + h\sum_i(\mathbf{x}(i) - \mathbf{d}(i))^2$, where $\mathbf{x}$, $\mathbf{d}$ are configurations, $\mathbf{x}(i), \mathbf{d}(i) \in \mathbb{R}$, $i, j \in \mathbb{Z}^d$. The configuration $\mathbf{d}$ is given and considered as observations. The 'couplings' $J_{i,j}$ are finite range. We use a version of the harness process to explicitly construct the unique infinite volume measure at finite temperature and to find the unique ground state configuration $\mathbf{m}$ corresponding to the Hamiltonian.

Keywords Non-homogeneous harmonic crystal · Harness process

1 Introduction

The harnesses were introduced by Hammersley [10] to model the behavior of a crystal and to introduce a multi-dimension version of a martingale. Let $P = (p(i, j), i, j \in \mathbb{Z}^d)$ be a homogeneous symmetric Markov transition matrix with $p(i, i) = 0$. A *harness* is a measure on $\mathbb{R}^{\mathbb{Z}^d}$ with the property that the mean height at i given the heights at all sites different of i is a P-weighted mean of the heights of the other sites. The *serial harness* is a Markov process on $\mathbb{R}^{\mathbb{Z}^d}$ updated at all discrete times at all sites by the rule: substitute the height at site i by a P-weighted mean of the neighbors plus a centered independent random variable (the noise). Hsiao [11] proposed a continuous-time version then called *harness process* in [7]. The heights are updated at Poisson epochs using the same rule as in the serial harness. If the noise is a centered Gaussian random variable, the reversible measure of the process is the

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harmonic crystal, that is, the Gibbs measure with Hamiltonian

$$H(\mathbf{x}) := \sum_{i,j} J_{i,j}(\mathbf{x}(i) - \mathbf{x}(j))^2 \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}^d}$, $\mathbf{x}(i)$ for $i \in \mathbb{Z}^d$ represents the height at site i and $J_{i,j} = p(i, j)$; the temperature $1/\beta$ is given by the variance of the noise.

We study a version of the harness process with a external local data term. We can think that each site $i \in \mathbb{Z}^d$ has an additional "neighbor" with a fixed height $\mathbf{d}(i)$, the data. The updating of the height at i involves the data $\mathbf{d}(i)$ in the averaging. This is a "heat bath" dynamics associated to the quadratic (Gaussian) Hamiltonian (2.2): at rate 1, the height at site i is substituted by a random height distributed with the conditional distribution associated to the Hamiltonian, given the heights at the other sites. We show ergodicity of the harness process with data $\mathbf{d} \in \mathbf{X}$, a set of configurations with a mild restriction on the growth, defined in (2.5); this extends the work of Hsiao [11, 12] who considered the case $\mathbf{d}(i) \equiv 0$. Ergodicity means that there exists a unique invariant measure for the process and that the process starting from measures concentrating mass on $\mathbf{X}$ converges to the invariant measure. The unique invariant measure is also reversible. We show that any infinite-volume Gibbs measure on $\mathbf{X}$ associated to the quadratic Hamiltonian (2.2) is invariant for the dynamics. This fact and ergodicity imply that there is only one infinite-volume Gibbs measure associated to the Hamiltonian. When the process is constructed in a finite subset Λ of $\mathbb{Z}^d$, the invariant measure is a harmonic crystal with external site-by-site field $\mathbf{d}$. As Λ grows to $\mathbb{Z}^d$ the harmonic crystals in Λ converge to the unique invariant measure for the harness process in $\mathbb{Z}^d$. If the data is flat (i.e. constant) then we are in the case of massive lattice models in quantum field theory which are well known (see [5, 8, 9]).

Shortly our method is as following. We slice the space-time configuration space $\Xi = \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}_+$ in pieces determined by the realizations of the Poisson processes governing the updating times. We show the convergence of the process almost surely for (almost) every slice. To this end we use Harris graphical construction of the process as a function of a space-time marked Poisson process of rate 1 on $\mathbb{Z}^d \times \mathbb{R}$. It is convenient to construct the process in an arbitrary interval $[s, t]$ to be able to take the limit as $s \to -\infty$ which is equivalent to take the limit as $t \to \infty$ in distribution. At each Poisson epoch, the value of the process at corresponding site is substituted by an average of the values of the process at the neighboring sites plus an independent noise. When the construction is explored backwards in time the value of the process at site i at time t can be expressed in function of the probabilities of a random walk running backwards in time *conditioned* on the space-time epochs of the Poisson process. The walk is killed at a rate related to the weight of the external data and when it hits the boundary. The value of the process at site i at time t starting at time s is expressed as a sum of four terms: the contributions given by (1) the noise, (2) the external data, (3) the boundary condition (in case the process is studied in a finite region) and (4) the initial condition. The noise contribution is a martingale with uniformly bounded second moments so it converges as $s \to -\infty$. The data contribution converges to a deterministic function $\mathbf{m}$, a harmonic function for a kernel associated to J and h. The boundary and initial contributions go to zero as the region grows to $\mathbb{Z}^d$ and the initial time s goes to $-\infty$, respectively.

In Sect. 2 we state the main three theorems. The first one shows the existence of the Harness process in infinite volume. The second one says that the Harness process with external data is ergodic, that its unique invariant measure coincides with the infinite-volume Gibbs measure and that there is a unique infinite volume Gibbs measure for this Hamiltonian. The third theorem shows that the harmonic function $\mathbf{m}$ is the unique function minimizing the Hamiltonian with the data term (ground state). In Sect. 3 we construct the process and state intermediate results for the finite and infinite process. In Sects. 4 and 5 we show that the finite process has as reversible measure the Gibbs measure in finite volume and that the space and time limits coincide. In Sect. 6 we show that the harmonic function $\mathbf{m}$ is the unique minimizer.

2 Main Results

Let $\Lambda \subset \mathbb{Z}^d$ be a finite set and define the Hamiltonian

$$H_\Lambda(\mathbf{x}) = \sum_{i,j:\{i,j\}\not\subset\Lambda^c} J_{i,j}(\mathbf{x}(i)-\mathbf{x}(j))^2 + h\sum_{i\in\Lambda}(\mathbf{x}(i)-\mathbf{d}(i))^2 \tag{2.2}$$

$\mathbf{x}, \mathbf{d} \in \mathbb{R}^{\mathbb{Z}^d}$, $J_{i,j}$ is a finite range pair potential, that is $J_{i,j} = 0$ if $|i-j| \ge R > 0$, $\mathbf{d}$ is a fixed configuration which can be taken as data and $h > 0$ is a fixed parameter. In fact $H_\Lambda = H_\Lambda(\cdot, \mathbf{d}, h)$ depends on $\mathbf{d}$ and h which are fixed and dropped from the notation unless necessary. Let $\mathbf{x}_\Lambda$ be the configuration $\mathbf{x}$ restricted to the set Λ and the superposition configuration $\mathbf{x}_\Lambda\mathbf{y}_{\Lambda^c}$ be given by $\mathbf{x}$ for sites in Λ and by $\mathbf{y}$ otherwise. Let $|\Lambda|$ be the number of sites in Λ. The family of measures $\{\mu_\Lambda(\cdot|\mathbf{y}_{\Lambda^c}), \Lambda \subset \mathbb{Z}^d, |\Lambda| < \infty, \mathbf{y} \in \mathbb{R}^{\mathbb{Z}^d}\}$, defined by

$$\mu_\Lambda(\mathrm{d}\mathbf{x}_\Lambda|\mathbf{y}_{\Lambda^c}) = \frac{e^{-H_\Lambda(\mathbf{x}_\Lambda\mathbf{y}_{\Lambda^c})}}{Z_\Lambda}\mu^0_\Lambda(\mathrm{d}\mathbf{x}_\Lambda), \tag{2.3}$$

where μ^0_Λ is the Lebesgue measure in $\mathbb{R}^\Lambda$ and Z_Λ is a normalizing constant, is called the specification associated to the Hamiltonian H. A measure μ defined on $\mathbb{R}^{\mathbb{Z}^d}$ is said to satisfy the Dobrushin–Lanford–Ruelle (DLR) equations if its conditioned distributions coincide with the specifications:

$$\mu(\mathrm{d}\mathbf{x}_\Lambda|\mathbf{x}_{\Lambda^c} = \mathbf{y}_{\Lambda^c}) = \mu_\Lambda(\mathrm{d}\mathbf{x}_\Lambda|\mathbf{y}_{\Lambda^c}) \tag{2.4}$$

for μ-almost all $\mathbf{y}$.

We construct the dynamics in a set of configurations with limited growth. Let

$$\mathbf{X} = \left\{\mathbf{x} \in \mathbb{R}^{\mathbb{Z}^d} : \sum_j |\mathbf{x}(j)|\alpha^{|j|/R} < \infty \text{ for all } i \in \mathbb{Z}^d\right\}. \tag{2.5}$$

Recall that R is the radius of the interactions. We introduce the *harness process* $\{\eta_t(i), i \in \mathbb{Z}^d\}$ on $\mathbf{X}$. Let $\alpha \in [0,1]$, $P = (p(i,j), i, j \in \mathbb{Z}^d)$ be a space homogeneous finite-range symmetric stochastic matrix (that is, $p(i,j) \ge 0, \sum_j p(i,j) = 1$ for all i, $p(i,i+j) = p(0,j)$, $p(0,j) = 0$ if $|j| \ge R > 0$ and $p(i,j) = p(j,i)$). Symmetry is not necessary to define the process but it is natural in this context: we relate later $p(i,j)$ with $J_{i,j}$ which is symmetric.

The generator of the process acts on locally finite continuous functions f by

$$\begin{aligned} Lf(\mathbf{x}) &= \sum_{k\in\mathbb{Z}^d} L_k f(\mathbf{x}) \\ &= \sum_{k\in\mathbb{Z}^d} \int G(\mathrm{d}x)[f(\alpha P_k\mathbf{x} + (1-\alpha)\mathbf{e}_k\mathbf{d}(k) + \mathbf{e}_k x) - f(\mathbf{x})], \end{aligned} \tag{2.6}$$

where $G(\mathrm{d}x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\mathrm{d}x$ is the standard Gaussian distribution (with zero mean and variance 1), called *noise* and $P_k\mathbf{x}$ is the configuration defined by

$$(P_k\mathbf{x})(j) = \begin{cases} \sum_{i\in\mathbb{Z}^d} p(k,i)\mathbf{x}(i) & \text{if } j=k, \\ \mathbf{x}(j) & \text{otherwise} \end{cases} \tag{2.7}$$

and $\mathbf{e}_k(j) = \mathbf{1}\{k=j\}$. In this process at rate 1 the height at site i is updated with a convex combination of the heights at the neighbors of i and the height $\mathbf{d}(i)$, plus an independent standard Gaussian variable. That is, if site i is updated at time t, the height at i is substituted by a random variable with the same law as

$$\alpha(P_i\eta_{t-})(i) + (1-\alpha)\mathbf{d}(i) + Z \tag{2.8}$$

where Z is a standard Gaussian independent of η_{t-}. The real variable (2.8) has law $\mu_{\{i\}}(\cdot|(\eta_{t-})_{\{i\}^c})$, the distribution of the $\{i\}$ coordinate given the values of η_{t-} in $\{i\}^c$ as defined in (2.3). Our dynamics coincides with the so called "heat bath".

The following three theorems are the aim of this work. The first one shows the existence of the process.

Theorem 2.1 *Assume* $\mathbf{d}\in\mathbf{X}$. *There exists a Markov process* (η_t) *on* $\mathbf{X}$ *with generator* L:

$$\lim_{u\to 0}\frac{1}{u}\mathsf{E}[f(\eta_{t+u}) - f(\eta_t)|\mathcal{F}_t] = Lf(\eta_t), \tag{2.9}$$

where $\mathcal{F}_t$ *is the sigma algebra generated by* $(\eta_s, s\le t)$, *the past of* η_s *up to time* t.

Theorem 2.1 is a consequence of a general existence result of Basis [1, 2]; we provide here an alternative construction.

Let $S(t)$ be the semigroup associated to L defined by $(S(t)f)(\eta) = \mathsf{E}(f(\eta_t)|\eta_0=\eta)$. It acts on measures by $\int f\mathrm{d}(\nu S(t)) = \int S(t)f\mathrm{d}\nu$. Our second result says that the harmonic crystal is the invariant measure for the harness process when $p(0,0)=0$ and establishes time and space limits.

Theorem 2.2 *Assume* $\mathbf{d}\in\mathbf{X}$ *and* $p(0,0)=0$. (i) *The following time and space limits exist and are identical. For any initial measure* ν *concentrating on* $\mathbf{X}$, *any boundary conditions* $\mathbf{y}\in\mathbf{X}$ *and any increasing sequence* $\Lambda\nearrow\mathbb{Z}^d$,

$$\lim_{t\to\infty}\nu S(t) = \lim_{\Lambda\nearrow\mathbb{Z}^d}\mu_\Lambda := \mu. \tag{2.10}$$

(ii) μ *is reversible for* L_k *for all* $k\in\mathbb{Z}^d$, *in particular it is reversible for* L. (iii) μ *is the unique measure in* $\mathbf{X}$ *satisfying the DLR equations for the specifications* (2.3) *with* $J_{i,j} = \alpha p(i,j)$ *and* $h = 1-\alpha$.

The theorem implies that the harness process is ergodic in $\mathbf{X}$: there exists a unique invariant measure μ and the process starting in $\mathbf{X}$ converges to μ. It also implies that for any boundary conditions in $\mathbf{X}$ the thermodynamic limit is unique (absence of phase transition).

The time convergence was proven by Hsiao [11] and [12] in the case $\mathbf{d}(k)\equiv 0$. The space convergence is contained in Spitzer [13] and Dobrushin [4], see also Caputo [3]. Our approach permits to construct simultaneously (coupling) realizations of the measures in all finite boxes and the infinite volume measure in such a way that the convergence is almost sure. Notice however that the uniqueness result is a consequence of the fact that there is only one infinite-volume invariant measure for the process and that any measure compatible with the DLR conditions is invariant. The space thermodynamical limit is not used in the proof of uniqueness.

The condition $p(0,0)=0$ is necessary to guarantee that the measure μ_Λ is invariant for the generator L_k.

In Statistical Mechanics it is natural to extend the specifications (2.3) to a family of measures μ_Λ^β defined by

$$\mu_\Lambda^\beta(\mathrm{d}\mathbf{x}_\Lambda|\mathbf{y}_{\Lambda^c}) = \frac{1}{Z_\Lambda^\beta} e^{-\beta H_\Lambda(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c})} \mu_\Lambda^0(\mathrm{d}\mathbf{x}_\Lambda), \tag{2.11}$$

for $\beta > 0$; β is called the inverse temperature. We consider in detail the case $\beta = 1$; the other cases reduce to this one using $\beta H_\Lambda(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c}, \mathbf{d}, h) = H_\Lambda(\mathbf{x}_\Lambda^\beta \mathbf{y}_{\Lambda^c}^\beta, \mathbf{d}^\beta, h)$, where $\mathbf{x}^\beta(i) = \sqrt{\beta}\mathbf{x}(i)$, etc.

When $\beta = \infty$ the randomness vanishes and μ_Λ^∞ is interpreted as the measure concentrating mass on configurations $\mathbf{x}_\Lambda$ minimizing $H_\Lambda(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c})$ for finite Λ. When $\Lambda = \mathbb{Z}^d$, we denote $H(\mathbf{x}) = H_{\mathbb{Z}^d}(\mathbf{x})$, an infinite sum only formally defined. In this case we need to give a sense to the word "minimizing". If $\tilde{\mathbf{x}}$ differs from $\mathbf{x}$ on a finite set of sites Λ, then the infinite sums defining $H(\mathbf{x})$ and $H(\tilde{\mathbf{x}})$ differ only on a finite number of summands. We define $H(\tilde{\mathbf{x}}) - H(\mathbf{x})$ as the difference of the corresponding different summands, that is, $H_\Lambda(\tilde{\mathbf{x}}_\Lambda \mathbf{x}_{\Lambda^c}) - H_\Lambda(\mathbf{x}_\Lambda \mathbf{x}_{\Lambda^c})$. We say that $\mathbf{x}$ *minimizes* $H(\cdot)$ if $H(\tilde{\mathbf{x}}) - H(\mathbf{x}) > 0$ for all $\tilde{\mathbf{x}}$ local modification of $\mathbf{x}$. Measures defined on $\mathbf{X}$ compatible with the specifications (2.3) with $\beta = \infty$ are called *ground states*; see Appendix B of van Enter, Fernández and Sokal [6] for details. The ground states are concentrated on minimizing configurations.

Let $\mathrm{K}(i,j)$ be the probability that the walk with rates αP (a walk killed at rate $1-\alpha$) is killed at site j when starting at site i. In the next theorem we show that a delta measure concentrating mass in the configuration given by the K-average of $\mathbf{d}$ is the unique ground state.

Theorem 2.3 *Assume* $p(0,0)=0$ *and that* $\mathbf{d} \in \mathbf{X}$ *and let* $\mathbf{m}$ *be the configuration given by*

$$\mathbf{m}(i) := \sum_j \mathrm{K}(i,j)\mathbf{d}(j) \quad \textit{for all } i \in \mathbb{Z}^d \tag{2.12}$$

then $\mathbf{m}$ *minimizes the Hamiltonian* H *and the delta measure concentrating mass on* $\mathbf{m}$ *is the unique ground state for specifications* (2.3) *in* $\mathbf{X}$.

Using the Kolmogorov Backwards equation for the walk with transitions αP killed at rate $1-\alpha$ and boundary conditions $\mathbf{d}$, we see that $\mathbf{m}$ satisfies the equation

$$\mathbf{m}(i) = \sum_{j \in \mathbb{Z}^d} \alpha p(i,j)\mathbf{m}(j) + (1-\alpha)\mathbf{d}(i). \tag{2.13}$$

That is, $\mathbf{m}$ is a harmonic function for a transition matrix associated to P and α in an extended graph with "boundary conditions" $\mathbf{d}$. The extended graph has vertices in $\mathbb{Z}^d \cup (\mathbb{Z}^d)^*$ where $(\mathbb{Z}^d)^*$ is a copy of $\mathbb{Z}^d$, and edges $\{(i,j) \in \mathbb{Z}^d \times \mathbb{Z}^d : |i-j| = 1\} \cup \{(i,i^*) : i \in \mathbb{Z}^d$ and i^* is the copy of i in $(\mathbb{Z}^d)^*\}$. The boundary conditions are fixed in $(\mathbb{Z}^d)^*$ equal to $\mathbf{d}$; $\mathbf{d}(i)$ is the value of the boundary condition at i^* and $(1-\alpha)$ is the weight of the edge (i,i^*). $\alpha p(i,j)$ is the weight of the edge (i,j).

3 Harris Graphical Construction

The proof of the above theorems are based on an adaptation of the Harris graphical construction for the harness process proposed by the first two authors in [7]. The core of the construction is a rate-one space-time Poisson process $\mathcal{N}$ on $\mathbb{Z}^d \times \mathbb{R}$. This can be thought of as a product of homogeneous one-dimensional Poisson process in $\mathbb{R}$, one for each $i \in \mathbb{Z}^d$. Space-time points in $\mathcal{N}$ are denoted (i,τ) and called *epochs*. To each $(i,\tau) \in \mathcal{N}$ we attach two independent *marks*: $\xi(i,\tau)$ and $\varphi(i,\tau)$, where $\xi(i,\tau)$ is a Gaussian random variable with zero mean and variance 1 called *noise* and $\varphi(i,\tau)$ are variables whose distribution is described later; these two families are iid and mutually independent and independent of $\mathcal{N}$. We denote P and E the probability and expectation induced by these Poisson processes with marks.

The Harness process is realized as a function of the Poisson epochs and the marks ξ (for the moment we do not use the marks φ) as follows. The height at each site i only changes at times $(i,s) \in \mathcal{N}$. Assuming that the configuration at time $s-$ is η_{s-} and $(i,s) \in \mathcal{N}$ is a Poisson epoch, then the configuration at time s at sites $k \neq i$ does not change ($\eta_s(k) = \eta_{s-}(k)$) and

$$\eta_s(i) = \alpha(P_i\eta_{s-})(i) + (1-\alpha)\mathbf{d}(i) + \xi(i,s) \tag{3.14}$$

where P_i is defined in (2.7). In other words, at the Poisson epoch (i,s), the height at i is substituted by a average of the other sites and the external data at i plus a Normal random variable independent of "everything" else. By a standard percolation argument this construction can be performed for small time intervals, so that $\mathbb{Z}^d$ is partitioned in non interacting pieces; we sketch it in the proof of (3.4) in Sect. 5 later. We give another construction based on a "dual" representation using the variables φ.

Let the variables $\varphi(i,\tau)$ be independent with law

$$\begin{aligned} &\mathsf{P}(\varphi(i,\tau) = j) = \alpha p(i,j), \\ &\mathsf{P}(\varphi(i,\tau) = i^*) = 1-\alpha, \end{aligned} \tag{3.15}$$

where i^* ($\notin \mathbb{Z}^d$) is the copy of i in $(\mathbb{Z}^d)^*$. The parameter $\alpha \in (0,1)$ is later chosen as in Theorem 2.2. Define a family of backward random walks indexed by (γ,t), the space-time starting point, as a deterministic function of the Poisson epochs $\mathcal{N}$ and the marks φ (here we do not use the marks ξ). Fix $t \in \mathbb{R}$ and $\gamma \in \mathbb{Z}^d \cup (\mathbb{Z}^d)^*$. For each $s < t$ we define $\sigma^\gamma_{[s,t]} \in \mathbb{Z}^d \cup (\mathbb{Z}^d)^*$ as the position of a random walk going backwards in time with initial position (at time t) $\sigma^\gamma_{[t,t]} = \gamma$ and evolving with the following rules. The walk does not move between Poisson epochs and for $s < t$, if at time $s+$ the walk is at site $i \in \mathbb{Z}^d$ and $(i,s) \in \mathcal{N}$, then at time s the walk jumps to the position $\varphi(i,s)$: for $s < t$,

$$\sigma^\gamma_{[s,t]} = \begin{cases} \varphi(i,s) & \text{if } \sigma^\gamma_{[s+,t]} = i \in \mathbb{Z}^d \text{ and } (i,s) \in \mathcal{N}, \\ \sigma^\gamma_{[s+,t]} & \text{if } \sigma^\gamma_{[s+,t]} = i^* \in (\mathbb{Z}^d)^* \text{ or } \sigma^\gamma_{[s+,t]} = i \in \mathbb{Z}^d \text{ and } (i,s) \notin \mathcal{N}. \end{cases} \tag{3.16}$$

We say that the walk $\sigma^\gamma_{[s,t]}$ is *absorbed* at $j \in \mathbb{Z}^d$ at time s if $\sigma^\gamma_{[u,t]} = j^*$ for $u \le s$ and $\sigma^\gamma_{[u,t]} \in \mathbb{Z}^d$ for $u \in (s,t]$. The family $((\sigma^\gamma_{[u,t]}, u \le t),\ t \in \mathbb{R}, \gamma \in \mathbb{Z}^d \cup (\mathbb{Z}^d)^*)$ is a function of $\mathcal{N}$ and φ, but we drop this dependence in the notation.

A key object in this analysis is the law of the walk σ *conditioned* on a realization of the Poisson epochs $\mathcal{N}$. For $i \in \mathbb{Z}^d$ let

$$b_{[u,t]}(i,j) = \mathsf{P}(\sigma^i_{[u,t]} = j \,|\, \mathcal{N}) \tag{3.17}$$

be the probability the walk σ starting at i at time t to be at j at time u *given* the Poisson epochs. The probabilities $b_{[u,t]}(i,j)$ are function of the Poisson epochs $\mathcal{N}$ and do *not* depend on φ. Analogously, for $(j,\tau)\in\mathcal{N}$ define the probability of absorption at time τ at site j given the Poisson epochs by

$$a_{[\tau,t]}(i,j^*) = \mathsf{P}(\sigma^i_{[\tau,t]} = j^*, \sigma^i_{[u,t]}\in\mathbb{Z}^d,\ \text{for } u\in(\tau,t]|\mathcal{N}). \tag{3.18}$$

Notice that

$$a_{[\tau,t]}(i,j^*) = (1-\alpha)b_{[\tau,t]}(i,j). \tag{3.19}$$

Define the $\mathbf{X}$-valued process $\eta_{[s,t]}$ in the time interval $[s,t]$ with $s\le t$ and initial condition $\mathbf{z}$ at time s by

$$\begin{aligned} \eta_{[s,s]} &\equiv \mathbf{z},\\ \eta_{[s,t]}(i) &= \sum_{(j,\tau)\in\mathcal{N}[s,t]} b_{[\tau,t]}(i,j)\xi(j,\tau) + a_{[\tau,t]}(i,j^*)\mathbf{d}(j) + \sum_{j\in\mathbb{Z}^d} b_{[s,t]}(i,j)\mathbf{z}(j), \end{aligned} \tag{3.20}$$

where $\mathcal{N}[s,t] = \{(j,\tau)\in\mathcal{N} : \tau\in[s,t]\}$ and recall $\xi(j,\tau)$ is the noise associated to the epoch (j,τ). Under this construction $\eta_{[s,t]}(i)$ is a function of $\mathcal{N}[s,t]$ and the corresponding noises ξ; it is an average determined by $\mathcal{N}$ of the noises ξ, the external field $\mathbf{d}$ and the initial condition $\mathbf{z}$ at time s. Our goal is to prove the following

3.1 *For each $s\in\mathbb{R}$ and $\mathbf{z}\in\mathbf{X}$ the process $(\eta_{[s,t]}, t\ge s)$ defined in (3.20) is well-defined. Namely, the sums in (3.20) are finite with probability 1 and $\eta_{[s,t]}\in\mathbf{X}$ for all $s<t$. Furthermore the process is Markovian with generator L given in (2.6) and initial condition $\mathbf{z}$ at time s.*

3.2 *For any configuration $\mathbf{z}\in\mathbf{X}$ and fixed $t\in\mathbb{R}$, the limit*

$$\lim_{s\to-\infty}\eta_{[s,t]}(i) := \eta_t(i) \tag{3.21}$$

exists with probability one and does not depend on $\mathbf{z}$. *The process $(\eta_t, t\in\mathbb{R})$ is a stationary Markov process with generator L given in (2.6).*

3.3 *Call μ the marginal law of η_t (which does not depend on t), then μ satisfies the DLR equations (2.4).*

3.4 *If $\widetilde{\mu}$ satisfies the DLR equations then $\widetilde{\mu}$ is reversible for the process $(\eta_{[s,t]}, t\ge s)$.*

3.5 *μ, the marginal law of η_t, is the unique infinite volume Gibbs measure on $\mathbf{X}$ for the specifications (2.3).*

In other words, the strategy of our proof is to construct a stationary process in infinite volume whose time marginal is the unique Gibbs measure for the Hamiltonian (2.2). Theorem 2.1 follows from 3.1.

Finite volume We start considering the Hammersley processes in finite volume $\Lambda \subset \mathbb{Z}^d$ with boundary conditions $\mathbf{y} \in \mathbf{X}$. The updates occurs at space-time Poisson epochs $(i, s) \in \mathcal{N}$ as follows:

$$\eta_s(i) = \alpha\left(\sum_{j\in\Lambda} p(i, j)\eta_{s-}(j) + \sum_{j\in\Lambda^c} p(i, j)\mathbf{y}(j)\right) + (1-\alpha)\mathbf{d}(i) + \xi(i, s). \tag{3.22}$$

That is, at the space-time Poisson epochs, the process substitutes the value at i by an average of the values at the other sites and the external data, the values outside Λ are kept fixed and given by the boundary configuration $\mathbf{y}$.

The construction of the finite harness process in Λ with boundary configuration $\mathbf{y}$ goes along the same lines as in infinite volume, the difference is that the probabilities b and a are computed for walks that are also absorbed at Λ^c.

Call Λ^* the copy of Λ in $(\mathbb{Z}^d)^*$. We define a family of backward random walks absorbed at $\Lambda^c \cup \Lambda^*$ indexed by $(\gamma, [s, t], \Lambda)$, with space-time starting point (γ, t) with $\gamma \in \Lambda$. The rules now are

$$\sigma^\gamma_{[s,t],\Lambda} = \begin{cases} \varphi(i, s) & \text{if } \sigma^\gamma_{[s+,t],\Lambda} = i \in \Lambda \text{ and } (i, s) \in \mathcal{N}, \\ \sigma^\gamma_{[s+,t],\Lambda} & \text{if } \sigma^\gamma_{[s+,t],\Lambda} \in (\mathbb{Z}^d)^* \cup \Lambda^c \text{ or } \sigma^\gamma_{[s+,t],\Lambda} = i \in \mathbb{Z}^d \text{ and } (i, s) \notin \mathcal{N}. \end{cases} \tag{3.23}$$

The only difference is that now the walk is absorbed at Λ^* *and* at Λ^c. If the walk starts at Λ, it will be absorbed either at Λ^* or at Λ^c. Calling

$$\mathcal{N}([s, t], \Lambda) = \{(j, \tau) \in \mathcal{N} : \tau \in [s, t], j \in \Lambda\},$$

the family $(\sigma^\gamma_{[u,t],\Lambda}, u \in [s, t]), t \in \mathbb{R}, \gamma \in \Lambda$ is a function of $\mathcal{N}([s, t], \Lambda)$ and the associated φ, but we drop this dependence in the notation. For $i \in \Lambda$ and $u < t$ define

$$b_{[u,t],\Lambda}(i, j) = \mathsf{P}(\sigma^i_{[u,t],\Lambda} = j|\mathcal{N}), \tag{3.24}$$

the transition probabilities for the walk absorbed at $(\mathbb{Z}^d)^*$ and Λ^c given the Poisson epochs. The probabilities of absorption at time τ at site $\gamma \in \Lambda^* \cup \Lambda^c$ given the Poisson epochs are defined by

$$a_{[\tau,t],\Lambda}(i, \gamma) = \mathsf{P}(\sigma^i_{[\tau,t],\Lambda} = \gamma, \sigma^i_{[u,t],\Lambda} \in \Lambda, \text{ for } u \in (\tau, t]|\mathcal{N}). \tag{3.25}$$

We define the $\mathbb{R}^\Lambda$-valued process $\eta_{[s,t],\Lambda}$ in the time interval $[s, t]$ with $s \le t$, initial condition $\mathbf{z} \in \mathbf{X}$ and boundary conditions $\mathbf{y} \in \mathbf{X}$ at time s by

$$\begin{aligned} \eta_{[s,s],\Lambda} &\equiv \mathbf{z}_\Lambda, \\ \eta_{[s,t],\Lambda}(i) &= \sum_{(j,\tau)\in\mathcal{N}([s,t],\Lambda)} \Bigg(b_{[\tau,t],\Lambda}(i, j)\xi(j, \tau) + a_{[\tau,t],\Lambda}(i, j^*)\mathbf{d}(j) \\ &\quad + \sum_{k\in\Lambda^c} a_{[\tau,t],\Lambda}(i, k)\mathbf{y}(k)\Bigg) + \sum_{j\in\Lambda} b_{[s,t],\Lambda}(i, j)\mathbf{z}_\Lambda(j), \quad \text{for } i \in \Lambda \end{aligned} \tag{3.26}$$

where $\mathcal{N}([s, t], \Lambda) = \{(j, \tau) \in \mathcal{N} : j \in \Lambda, \tau \in [s, t]\}$, the value $\xi(j, \tau)$ is the Gaussian random variable associated to the Poisson epoch $(j, \tau) \in \mathcal{N}$.

With the above construction $\eta_{[s,t],\Lambda}(i)$ is an average determined by $\mathcal{N}$ (through the weights a and b) of the noise ξ, the external field $\mathbf{d}$, the boundary configuration $\mathbf{y}$ and the initial condition $\mathbf{z}$ at time s. We drop these dependences in the notation.

We prove the following facts about the finite process:

3.6 *For each $s \in \mathbb{R}$ the process $(\eta_{[s,t],\Lambda}, t \geq s)$ is Markov with initial condition $\mathbf{z}_\Lambda$, boundary conditions $\mathbf{y}_{\Lambda^c}$ and generator*

$$
\begin{aligned}
L_\Lambda f(\mathbf{x}_\Lambda) &= \sum_{k\in\Lambda} L_k f(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c}) \\
&= \sum_{k\in\Lambda} \int G(\mathrm{d}x)[f([\alpha P_k(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c}) + (1-\alpha)\mathbf{e}_k \mathbf{d}(k) + \mathbf{e}_k x]_\Lambda) - f(\mathbf{x}_\Lambda)], \quad (3.27)
\end{aligned}
$$

where f is a bounded function depending only on coordinates in Λ.

3.7 *The measure $\mu_\Lambda(\cdot|\mathbf{y}_{\Lambda^c})$ given in 2.3 is reversible for the process $(\eta_{[s,t],\Lambda}, t \geq s)$ for each fixed $s \in \mathbb{R}$.*

3.8 *For any configuration $\mathbf{z} \in \mathbf{X}$ and fixed $t \in \mathbb{R}$, the limit*

$$\lim_{s\to-\infty} \eta_{[s,t],\Lambda}(i) =: \eta_{t,\Lambda}(i) \quad (3.28)$$

exists with probability one and does not depend on $\mathbf{z}$. *The process $(\eta_{t,\Lambda}, t \in \mathbb{R})$ is stationary with time-marginal $\mu_\Lambda(\cdot|\mathbf{y}_{\Lambda^c})$, which is the unique invariant measure for the process.*

3.9 *For any configuration $\mathbf{y} \in \mathbf{X}$ and fixed $t \in \mathbb{R}$, the limit*

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \eta_{t,\Lambda}(i) = \eta_t(i) \quad (3.29)$$

holds with probability one and does not depend on $\mathbf{y}$.

Theorem 2.2 follows from 3.2 to 3.9.

4 Finite Volume

In this section we prove 3.6, 3.7 and 3.8 for the finite process $\eta_{t,\Lambda}$.

Generator

Proof of 3.6 From (3.26) it follows that

$$
\begin{aligned}
\eta_{[s,u+h],\Lambda}(i) = \mathbf{1}\{|\mathcal{N}([u,u+h],\{i\})| = 0\}\eta_{[s,u],\Lambda}(i) + \mathbf{1}\{|\mathcal{N}([u,u+h],\{i\})| = 1\} \\
\times \Bigg(\alpha \sum_{j\in\Lambda} p(i,j)\eta_{[s,u],\Lambda}(j) + \alpha \sum_{j\in\Lambda^c} p(i,j)\mathbf{y}(j) \\
+ (1-\alpha)\mathbf{d}(i) + \xi(\tau,i)\Bigg) + \text{other terms} \quad (4.30)
\end{aligned}
$$

where $\tau \in (u,u+h)$, $\{|\mathcal{N}([u,u+h],\{i\})| = k\}$ is the event "there are exactly k Poisson epochs in $[u,u+h] \times \{i\}$" and the "other terms" are related to the presence of more than one Poisson epoch in $U(i) \times [u,u+h]$ which has probability of order h^2, where $U(i)$ is the cube centered at i with side R. The independence properties of the Poisson process and (4.30) show that the process is Markovian with generator (3.27). ☐

Invariant Measure

Proof of 3.7 Let $p(0,0)=0, \quad 0<h<1$ and $J_{i,j}=(1-h)p(i,j)$ and $h=1-\alpha$. We prove that μ_Λ defined by (2.3) on $\mathbb{R}^\Lambda$ is reversible for each one of the generators L_k defined in (3.27), that is, for $f,g:\mathbb{R}^\Lambda\to\mathbb{R}$,

$$\mu_\Lambda(gL_kf)=\mu_\Lambda(fL_kg). \tag{4.31}$$

For a fixed configuration $\mathbf{z}$ and site k let $k\in\Lambda$, let

$$\bar{\mathbf{z}}(k)=\sum_{i:\,i\neq k}J_{k,i}\mathbf{z}(i)+h\mathbf{d}(k). \tag{4.32}$$

Let $\mathbf{z}=\mathbf{x}_\Lambda\mathbf{y}_{\Lambda^c}$. A direct calculation yields

$$R_k:=\sum_{i:\,i\neq k}J_{k,i}(\mathbf{z}(i)-\mathbf{z}(k))^2+h(\mathbf{z}(k)-\mathbf{d}(k))^2 \tag{4.33}$$

$$=(\mathbf{z}(k)-\bar{\mathbf{z}}(k))^2+\sum_{i:\,i\neq k}J_{k,i}(\mathbf{z}(i)-\bar{\mathbf{z}}(k))^2+h(\bar{\mathbf{z}}(k)-\mathbf{d}(k))^2. \tag{4.34}$$

It follows that the conditional law at coordinate k given the heights at the other coordinates is a Gaussian with mean $\bar{\mathbf{z}}(k)$:

$$\mu_{\{k\}}(\mathrm{d}z|\mathbf{z}_{\{k\}^c})=\frac{1}{\sqrt{2\pi}}e^{-(z-\bar{\mathbf{z}}(k))^2}\mathrm{d}z \tag{4.35}$$

so that the updating at site k is done with the conditional distribution given the heights at the other sites. This implies reversibility; to show it we use the notations

$$E^k(\mathbf{z})=\exp\left\{-\sum_{i,j:\,i,j\neq k}J_{i,j}(\mathbf{z}(i)-\mathbf{z}(j))^2-\sum_{i:\,i\neq k}h(\mathbf{z}(i)-\mathbf{d}(i))^2\right\},$$

$$E_k(\mathbf{z},\mathbf{z}(k))=\exp\left\{-\sum_{i:\,i\neq k}J_{k,i}(\mathbf{z}(i)-\mathbf{z}(k))^2-h(\mathbf{z}(k)-\mathbf{d}(k))^2\right\}.$$

We represent the left-hand side in (4.31) as

$$\mu_\Lambda(gL_kf)=S_k^1-S_k^2,$$

where

$$S_k^1:=\frac{1}{Z_\Lambda}\int\prod_{i\in\Lambda}\mathrm{d}\mathbf{z}(i)E^k(\mathbf{z})E_k(\mathbf{z},\mathbf{z}(k))$$
$$\times\int_{\mathbb{R}}\mathrm{d}xe^{-x^2}g(\mathbf{z})f((1-h)P_k\mathbf{z}+h\mathbf{d}(k)+\mathbf{e}_kx),$$
$$S_k^2:=\frac{\sqrt{\pi}}{Z_\Lambda}\int\prod_{i\in\Lambda}\mathrm{d}\mathbf{z}(i)E^k(\mathbf{z})E_k(\mathbf{z},\mathbf{z}(k))g(\mathbf{z})f(\mathbf{z}).$$

Using (4.34), we obtain

$$S_k^1 = \frac{1}{Z_\Lambda}\int \prod_{i\in\Lambda} \mathrm{d}\mathbf{z}(i) E^k(\mathbf{z}) E_k(\mathbf{z},\bar{\mathbf{z}}(k)) \exp\{-(\mathbf{z}(k)-\bar{\mathbf{z}}(k))^2\}$$
$$\times \int_{\mathbb{R}} \mathrm{d}x e^{-x^2} g(\mathbf{z}) f((1-h)P_k\mathbf{z} + h\mathbf{d}(k) + \mathbf{e}_k x).$$

Next we change the variables as follows

$$\mathbf{v}(i) = \begin{cases} \bar{\mathbf{z}}(k) + x, & \text{if } i = k; \\ \mathbf{v}(i) = \mathbf{z}(i), & \text{otherwise;} \end{cases}$$

and

$$y = \mathbf{z}(k) - \bar{\mathbf{z}}(k).$$

Observe that $\bar{\mathbf{v}}(k) = \bar{\mathbf{z}}(k)$. Therefore $\mathbf{z}(i) - \bar{\mathbf{z}}(k) = \mathbf{v}(i) - \bar{\mathbf{v}}(k)$ for any $i \neq k$. Using first the above substitution and then relation (4.34) again yields

$$S_k^1 = \frac{1}{Z_\Lambda}\int \prod_{i\in\Lambda} \mathrm{d}\mathbf{v}(i) E^k(\mathbf{v}) E_k(\mathbf{v},\bar{\mathbf{v}}(k)) \exp\{-(\mathbf{v}(k)-\bar{\mathbf{v}}(k))^2\}$$
$$\times \int_{\mathbb{R}} \mathrm{d}y e^{-y^2} g((\bar{\mathbf{v}}(k)+y)\mathbf{v}_{\Lambda\setminus k}) f(\mathbf{v})$$
$$= \frac{1}{Z_\Lambda}\int \prod_{i\in\Lambda} \mathrm{d}\mathbf{v}(i) E^k(\mathbf{v}) E_k(\mathbf{v},\mathbf{v}(k))$$
$$\times \int_{\mathbb{R}} \mathrm{d}y e^{-y^2} g((1-h)P_k\mathbf{v} + h\mathbf{d}(k) + \mathbf{e}_k y) f(\mathbf{v}).$$

We thus obtain $\mu_\Lambda(fL_kg) = S_k^1 - S_k^2 = \mu_\Lambda(gL_kf)$. □

Limiting Distributions

Proof of 3.8 We consider separately the four terms in (3.26). Define

$$A_{[s,t],\Lambda}(i) := \sum_{(j,\tau)\in\mathcal{N}([s,t],\Lambda)} b_{[\tau,t],\Lambda}(i,j)\xi(j,\tau), \tag{4.36}$$

$$B_{[s,t],\Lambda}(i) := \sum_{(j,\tau)\in\mathcal{N}([s,t],\Lambda)} a_{[\tau,t],\Lambda}(i,j^*)\mathbf{d}(j), \tag{4.37}$$

$$C_{[s,t],\Lambda}(i) := \sum_{(j,\tau)\in\mathcal{N}([s,t],\Lambda)} \sum_{k\in\Lambda^c} a_{[\tau,t],\Lambda}(i,k)\mathbf{y}(k), \tag{4.38}$$

$$D_{[s,t],\Lambda}(i) := \sum_{j\in\mathbb{Z}^d} b_{[s,t],\Lambda}(i,j)\mathbf{z}(j). \tag{4.39}$$

Since the variables ξ are independent of the variables b, $(A_{[t-u,t],\Lambda}(i), u \geq 0)$ is a martingale for the sigma field $\mathcal{F}_u$ generated by $\mathcal{N}([t-u,t],\Lambda)$ and the associated Gaussian variables ξ.

The variance at time u is given by

$$\mathsf{E}(A_{[t-u,t],\Lambda}(i))^2 = \mathsf{E} \sum_{(j,\tau)\in\mathcal{N}([t-u,t],\Lambda)} (b_{[\tau,t],\Lambda}(i,j))^2 \tag{4.40}$$

by conditioning on the Poisson epochs and deleting the cross terms by independence of the different ξ. The last expression is bounded by

$$\le \mathsf{E} \sum_{(j,\tau)\in\mathcal{N}([t-u,t],\Lambda)} b_{[\tau,t],\Lambda}(i,j) = \sum_j \int_0^u e^{-(1-\alpha)r} Q^r_\Lambda(i,j)\mathrm{d}r \le \frac{1}{1-\alpha} \tag{4.41}$$

uniformly in Λ, where $Q^r_\Lambda(i,j)$ is the probability for a continuous time random walk with rates p absorbed at Λ^c starting at i to be at j by time r. The factor $e^{-(1-\alpha)r}$ corresponds to the probability that the walk is not killed in the time interval $[t, t-r]$. The Martingale convergence theorem implies that

$$\lim_{u\to\infty} A_{[t-u,t],\Lambda}(i) = A_{t,\Lambda}(i) \quad \text{a.s.} \tag{4.42}$$

The process $(A_{t,\Lambda}, t\in\mathbb{R})$ is Markov, stationary with generator

$$\sum_k L^A_k f(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c}) = \sum_k \int G(\mathrm{d}x)[f(\alpha P_k(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c} + \mathbf{e}_k x)) - f(\mathbf{x}_\Lambda \mathbf{y}_{\Lambda^c})], \tag{4.43}$$

and $\mathsf{E}(A_{t,\Lambda}(i))^2 \le (1-\alpha)^{-1}$.

We show now that the limit in (4.37) is given by

$$\lim_{u\to\infty} B_{[t-u,t],\Lambda}(i) = \sum_{j\in\Lambda} K_\Lambda(i,j)\mathbf{d}(j) =: \mathbf{m}_\Lambda(i) \tag{4.44}$$

where $K_\Lambda(i,j)$ is the probability the random walk to be killed at site j, if $j\in\Lambda$, or to be absorbed, if $j\in\Lambda^c$. As in (2.13), $\mathbf{m}_\Lambda$ satisfies

$$\mathbf{m}_\Lambda(i) = \sum_{k\in\Lambda} \alpha p(i,k)\mathbf{m}_\Lambda(k) + (1-\alpha)\mathbf{d}(i). \tag{4.45}$$

For $v>u$, the following "Markov property" holds

$$B_{[t-v,t],\Lambda}(i) := B_{[t-u,t],\Lambda}(i) + \sum_{k\in\Lambda} b_{[t-u,t],\Lambda}(i,k) B_{[t-v,t-u],\Lambda}(k). \tag{4.46}$$

Applying (4.45) to each Poisson epoch (j,τ) we have

$$\mathbf{m}_\Lambda(i) = B_{[t-u,t],\Lambda}(i) + \sum_{k\in\Lambda} b_{[t-u,t],\Lambda}(i,k)\mathbf{m}_\Lambda(k) \tag{4.47}$$

which implies that $\mathbf{m}_\Lambda$ is invariant for this dynamics. For fixed s the process $(B_{[s,t],\Lambda}, t\ge s)$ on $\mathbb{R}^\Lambda$ is Markov with generator

$$\sum_{k\in\Lambda} L^B_k f(\mathbf{x}_\Lambda) = \sum_{k\in\Lambda} [f([\alpha P_k(\mathbf{x}_\Lambda \mathbf{o}_{\Lambda^c}) + (1-\alpha)\mathbf{e}_k \mathbf{d}(k)]_\Lambda) - f(\mathbf{x}_\Lambda)], \tag{4.48}$$

where the $\mathbf{o}$ is the "all zero" configuration: $\mathbf{o}(i) \equiv 0$. Finally, using (4.47),

$$|B_{[t-u,t],\Lambda}(i) - \mathbf{m}_\Lambda(i)| \le \sum_k b_{[t-u,t],\Lambda}(i,k)|\mathbf{m}_\Lambda(k)|. \tag{4.49}$$

By (4.41) the right hand side converges to zero as $u \to \infty$ uniformly in Λ, proving (4.44).

A similar argument shows that

$$\lim_{u\to\infty} C_{[t-u,t],\Lambda}(i) = \sum_{j\in\Lambda^c} \mathrm{K}_\Lambda(i,j)\mathbf{y}(j) =: \mathbf{r}_\Lambda(i). \tag{4.50}$$

Finally, by (4.41),

$$\lim_{u\to\infty} D_{[t-u,t],\Lambda}(i) = 0. \tag{4.51}$$

The limits (4.42), (4.44), (4.50) and (4.51) show (3.28). The resulting limit is

$$\eta_{t,\Lambda} = A_{t,\Lambda} + \mathbf{m}_\Lambda + \mathbf{r}_\Lambda. \tag{4.52}$$

By construction the law of this limit does not depend on t. One proves that the process $(\eta_{t,\Lambda}, t \in \mathbb{R})$ is Markov like in 3.6. Since the limit (4.52) does not depend on the initial configuration $\mathbf{z}$, this shows that the process $(\eta_{t,\Lambda})$ has a unique invariant measure. Since by 3.8 $\mu_\Lambda(\cdot|\mathbf{y}_{\Lambda^c})$ is invariant for this process, for each $t \in \mathbb{R}$ the marginal law of $\eta_{t,\Lambda}$ is $\mu_\Lambda(\cdot|\mathbf{y}_{\Lambda^c})$. □

5 Infinite Volume and Thermodynamic Limit

Existence of the Process

Proof of 3.1 The fact that the sums in (3.20) are finite follows immediately from the finite range condition on p. The proof that the dynamics is the harness process follows then from an argument similar to the one in 3.6. □

Limiting Stationary Processes Here we prove 3.2 and 3.9. To prove 3.2 we consider separately the three terms in (3.20). Define

$$A_{[s,t]}(i) := \sum_{(j,\tau)\in\mathcal{N}([s,t])} b_{[\tau,t]}(i,j)\xi(j,\tau), \tag{5.53}$$

$$B_{[s,t]}(i) := \sum_{(j,\tau)\in\mathcal{N}([s,t])} a_{[\tau,t]}(i,j^*)\mathbf{d}(j), \tag{5.54}$$

$$D_{[s,t]}(i) := \sum_{j\in\mathbb{Z}^d} b_{[s,t]}(i,j)\mathbf{z}(j). \tag{5.55}$$

Lemma 5.1 *Under the hypothesis of Theorem 2.2,*

$$\lim_{u\to\infty} A_{[t-u,t]}(i) = A_t(i) \tag{5.56}$$

and $(A_t, t \in \mathbb{R})$ *is a stationary Markov process with generator*

$$\sum_k L_k^A f(\mathbf{x}) = \sum_k \int G(\mathrm{d}x)[f(\alpha P_k \mathbf{x} + \mathbf{e}_k x) - f(\mathbf{x})], \tag{5.57}$$

with $\mathsf{E}(A_t(i))^2 \le (1-\alpha)^{-1}$. *Furthermore,*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} A_{t,\Lambda}(i) = A_t(i). \tag{5.58}$$

Proof As in the proof of (4.42), $(A_{[t-u,t]}(i), u \ge 0)$ is a martingale with uniformly bounded second moments. Hence the limit (5.56) exists. Since $b_{[\tau,t],\Lambda}(i,j) \nearrow b_{[\tau,t]}(i,j)$,

$$|A_t(i) - A_{t,\Lambda}(i)| \le \sum_{(j,\tau)\in \mathcal{N}[-\infty,t]} (b_{[\tau,t]}(i,j) - b_{[\tau,t],\Lambda}(i,j))|\xi_{(j,\tau)}|. \tag{5.59}$$

The sum is finite because the variance of $A_t(i)$ is bounded and it converges to zero by monotone convergence.

The process $(A_t, t \in \mathbb{R})$ is stationary by construction. It is Markov with generator (5.57) as a consequence of (5.58) and 3.6. □

Lemma 5.2 *If* $\mathbf{d} \in \mathbf{X}$ *then* $|\mathbf{m}(i)| < \infty$ *for all* $i \in \mathbb{Z}^d$ *and*

$$\lim_{u\to\infty} B_{[t-u,t]}(i) = \mathbf{m}(i) \tag{5.60}$$

and

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \mathbf{m}_\Lambda(i) = \mathbf{m}(i). \tag{5.61}$$

Proof Remark that $\mathrm{K}(i,j) \le \alpha^{[\frac{|i-j|}{R}]}$ because at least $[\frac{|i-j|}{R}]+1$ jumps are needed to achieve the point j from i and during the time $\sigma^i_{[-\infty,t]}$ is not absorbed.

$$|\mathbf{m}(i)| \le \sum_j \mathrm{K}(i,j)|\mathbf{d}(j)| \le \sum_j \alpha^{\left[\frac{|i-j|}{R}\right]}|\mathbf{d}(j)| < \infty \tag{5.62}$$

by the definition of $\mathbf{X}$. The proof of (5.60) works as the proof of (4.44).

On the other hand $\mathrm{K}_\Lambda(i,j) \nearrow \mathrm{K}(i,j)$, hence

$$|\mathbf{m}(i) - \mathbf{m}_\Lambda(i)| \le \sum_j (\mathrm{K}(i,j) - \mathrm{K}_\Lambda(i,j))|\mathbf{d}(j)| \to 0 \tag{5.63}$$

by monotonic convergence. □

Lemma 5.3 *If* $\mathbf{y} \in \mathbf{X}$ *then for* $\mathbf{r}_\Lambda$ *defined in* (4.50),

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \mathbf{r}_\Lambda(i) = 0. \tag{5.64}$$

Proof As in the proof of the previous lemma,

$$|\mathbf{r}_\Lambda(i)| \le \sum_{j\in\Lambda^c} \alpha^{\left[\frac{|i-j|}{R}\right]} |\mathbf{y}(j)| \tag{5.65}$$

which converges to zero as $\Lambda \nearrow \mathbb{Z}^d$ because the sum is finite by definition of $\mathbf{X}$. □

Proof of 3.2 Calling $\eta_t =: A_t + \mathbf{m}$, the convergence follows from (5.56), (5.60) and (4.51) (which holds uniformly in Λ). □

Proof of 3.9 Since

$$\eta_{t,\Lambda} = A_{t,\Lambda} + \mathbf{m}_\Lambda + \mathbf{r}_\Lambda \quad \text{and} \quad \eta_t = A_t + \mathbf{m}, \tag{5.66}$$

the convergence follows from (5.58), (5.61) and (5.64). □

Limiting Process Satisfies DLR Conditions

Proof of 3.3 Here we use the thermodynamic limit. The measures $\mu_\Lambda(\cdot|\mathbf{y}_{\Lambda^c})$ are compatible with the DLR conditions for subsets of Λ. By (3.29) the measure μ is the limit as Λ increases to $\mathbb{Z}^d$ of $\mu_\Lambda(\cdot|\mathbf{y}_{\Lambda^c})$, independently of $\mathbf{y}\in\mathbf{X}$. Hence, also μ is compatible with the DLR conditions. □

DLR Measures Are Invariant

Proof of 3.4 This proof does not use the thermodynamic limit. We first prove that $\widetilde{\mu}$ is invariant. Let η be a configuration sampled with $\widetilde{\mu}$. Let $\overline{\eta}(k)$ be defined in (4.32). Let Z be a standard Gaussian variable independent of η. Since $\widetilde{\mu}$ satisfies the DLR equations, a computation like in (4.34) shows that the variable

$$\overline{\eta}(k) + Z \quad \text{has law} \quad \widetilde{\mu}(\mathbf{x}(k)\in\cdot|\mathbf{x}_{\{k\}^c} = \eta_{\{k\}^c})$$

and the configuration

$$\theta(k,\eta,Z) = \eta - \delta_{\eta(k)} + \delta_{\overline{\eta}(k)+Z}$$

obtained by substituting the value at k by $\overline{\eta}_k + Z$ has law $\widetilde{\mu}$. The Harness dynamics does the same substitution with the Gaussian noises ξ at the updating epochs; see (3.14). This means that after each Poisson epoch, the distribution of the updated configuration is the same as the one just before the updating. But since there are infinitely many sites, there is no "first Poisson epoch" to apply the rule and show 3.4 directly. To overcome the difficulty consider a time t small enough such that $\mathbb{Z}^d$ is partitioned in finite sets Λ_ℓ, $\ell = 1, 2, \ldots$ (depending on the Poisson epochs in the interval $[0,t]$) in such a way that if there is an epoch at time $s\in[0,t]$ at site $i\in\Lambda_\ell$, then $\{j\in\mathbb{Z}^d : ||i-j|| \le R\} \subset \Lambda_\ell$, where we recall R is such that $p(0,j) = 0$ if $||j|| \ge R$. A standard percolation argument shows that this is possible.

Enumerate the epochs as follows. Start ordering chronologically the epochs in Λ_1; continue with the epochs in Λ_2 and so on. Under this ordering, if $n(i,s)$ is the label of epoch (i,s) then $n(i,s) < n(i',s')$ if $s<s'$ or if $i\in\Lambda_\ell$, $i' = \Lambda_{\ell'}$ and $\ell<\ell'$. Call (i_n,s_n) the nth Poisson epoch in this labeling and $\xi_n = \xi(i_n,s_n)$ the associated Gaussian variable. Choose η with law $\widetilde{\mu}$ and define inductively $\eta_0 = \eta$ and for $n\ge 1$,

$$\eta_n = \theta(i_n,\eta_{n-1},\xi_n).$$

By the previous considerations η_n has law $\widetilde{\mu}$ for all n. By the definition of Λ_ℓ, the updating of sites in Λ_ℓ in the time interval $[0,t]$ depends on the initial configuration η only through the values at sites in Λ_ℓ. On the other hand, for any cylinder function with support on a finite set Λ, there exists an $n(\Lambda,\mathcal{N})<\infty$ such that $\Lambda\subset\{i_1,\dots,i_{n(\Lambda,\mathcal{N})}\}$ and $\int\widetilde{\mu}(\mathrm{d}\eta)\mathsf{E}(f(\eta_{n(\Lambda,\mathcal{N})})|\mathcal{N})=\widetilde{\mu}f$ for almost all $\mathcal{N}$. Hence

$$\widetilde{\mu}S(t)f=\int\widetilde{\mu}(\mathrm{d}\eta)\mathsf{E}(\mathsf{E}(f(\eta_{n(\Lambda,\mathcal{N})})|\mathcal{N}))=\mathsf{E}\Big(\int\widetilde{\mu}(\mathrm{d}\eta)\mathsf{E}(f(\eta_{n(\Lambda,\mathcal{N})})|\mathcal{N})\Big)=\widetilde{\mu}f.$$

This shows invariance of $\widetilde{\mu}$.

A similar argument shows reversibility. Going backwards in time, the law of $\eta_{n-1}(i_n)-\bar{\eta}_n(i_n)$ is a standard Gaussian variable independent of η_n. The construction can be done using the same Poisson epochs and partition (Λ_ℓ). □

6 Zero Temperature

Proof of Theorem 2.3 We need to show that the configuration $\mathbf{m}$ defined in (2.12) is compatible with the specifications (2.3) at zero temperature. Let $\Lambda\subset\mathbb{Z}^d$ be a finite volume. We want to show that $\mathbf{m}_\Lambda$ given by (2.12) is the $\mathbf{x}_\Lambda$ which minimizes

$$\begin{aligned}H(\mathbf{x}_\Lambda\mathbf{m}_{\Lambda^c})&=\alpha\sum_{i,j\in\Lambda}p(i,j)(\mathbf{x}(i)-\mathbf{x}(j))^2+\alpha\sum_{\substack{i\in\Lambda\\k\in\Lambda^c}}p(i,k)(\mathbf{x}(i)-\mathbf{m}(k))^2\\&\quad+(1-\alpha)\sum_{i\in\Lambda}(\mathbf{x}(i)-\mathbf{d}(i))^2.\end{aligned}\tag{6.67}$$

The Hamiltonian (6.67) can be rewritten as

$$\begin{aligned}H(\mathbf{x}_\Lambda\mathbf{m}_{\Lambda^c})&=\alpha\sum_{i\in\Lambda}\sum_{k\in\mathbb{Z}^d}p(i,k)(\mathbf{x}(i)-\mathbf{m}(k))^2+(1-\alpha)\sum_{i\in\Lambda}(\mathbf{x}(i)-\mathbf{d}(i))^2\\&\quad+2\alpha\sum_{i,j\in\Lambda}p(i,j)[-(\mathbf{x}(i)-\mathbf{m}(j))(\mathbf{x}(j)-\mathbf{m}(i))\\&\quad+(\mathbf{x}(i)-\mathbf{m}(j))(\mathbf{m}(j)-\mathbf{m}(i))-(\mathbf{x}(j)-\mathbf{m}(i))(\mathbf{m}(j)-\mathbf{m}(i))\\&\quad+(\mathbf{m}(j)-\mathbf{m}(i))^2].\end{aligned}\tag{6.68}$$

Let

$$r_i=\mathbf{m}(i)^2-\Big[\alpha\sum_{k\in\mathbb{Z}^d}p(i,k)\mathbf{m}(k)^2+(1-\alpha)\mathbf{d}(i)^2\Big]$$

then

$$\begin{aligned}H(\mathbf{x}_\Lambda\mathbf{m}_{\Lambda^c})&=\sum_{i\in\Lambda}(\mathbf{x}(i)-\mathbf{m}(i))^2-\sum_{i\in\Lambda}r_i\\&\quad+2\alpha\sum_{i,j\in\Lambda}p(i,j)[-(\mathbf{x}(i)-\mathbf{m}(j))(\mathbf{x}(j)-\mathbf{m}(i))\end{aligned}$$

$$+ (\mathbf{x}(i) - \mathbf{m}(j))(\mathbf{m}(j) - \mathbf{m}(i)) - (\mathbf{x}(j) - \mathbf{m}(i))(\mathbf{m}(j) - \mathbf{m}(i)) + (\mathbf{m}(j) - \mathbf{m}(i))^2].$$

To find the configuration $\mathbf{x}_\Lambda$ minimizing (6.67) we take the derivatives for $i \in \Lambda$:

$$\frac{\partial H}{\partial \mathbf{x}(i)} = 2(\mathbf{x}(i) - \mathbf{m}(i)) + \alpha \sum_{k\in\Lambda} p(i,k)[-2(\mathbf{x}(k) - \mathbf{m}(i)) + 2(\mathbf{m}(k) - \mathbf{m}(i))].$$

Then we obtain the following system of the equations

$$\mathbf{x}(i) - \alpha \sum_{k\in\Lambda} p(i,k)\mathbf{x}(k) = \mathbf{m}(i) - \alpha \sum_{k\in\Lambda} p(i,k)\mathbf{m}(k). \tag{6.69}$$

It is clear that $\mathbf{x}(i) = \mathbf{m}(i)$ is a solution of (6.69). Since H_Λ is convex, $\mathbf{m}$ is the minimum. Since H_Λ is a second order polynomial the minimum is unique. This shows that $\mathbf{m}$ belongs to the support of a ground state measure.

Uniqueness Let $\widetilde{\mathbf{m}} \in \mathbf{X}$ be a minimizer of H. Then $\widetilde{\mathbf{m}}(i) = \alpha \sum_j p(i,j)\widetilde{\mathbf{m}}(j) + (1-\alpha)\mathbf{d}(j)$ minimizes $H(\mathbf{x}_{\{i\}}\widetilde{\mathbf{m}}_{\{i\}^c})$ for each $i \in \mathbb{Z}^d$. So $\widetilde{\mathbf{m}}$ is invariant for the dynamics

$$\sum_{k\in\mathbb{Z}^d} L_k^B f(\mathbf{x}) = \sum_{k\in\mathbb{Z}^d} [f(\alpha P_k(\mathbf{x}) + (1-\alpha)\mathbf{e}_k\mathbf{d}(k)) - f(\mathbf{x})], \tag{6.70}$$

because this dynamics chooses a site i at random times and substitutes the height at i with the value minimizing $H(\mathbf{x}_{\{i\}}\widetilde{\mathbf{m}}_{\{i\}^c})$. The graphical construction of this process with initial measure $\widetilde{\mathbf{m}}$ and the invariance of $\widetilde{\mathbf{m}}$ gives

$$\widetilde{\mathbf{m}}(i) = B_{[t-u,t]}(i) + \sum_{k\in\mathbb{Z}^d} b_{[t-u,t]}(i,k)\widetilde{\mathbf{m}}(k). \tag{6.71}$$

Subtracting this equation from (4.47) with $\Lambda = \mathbb{Z}^d$ we get

$$|\mathbf{m}(i) - \widetilde{\mathbf{m}}(i)| \le \sum_{k\in\mathbb{Z}^d} b_{[t-u,t]}(i,k)|\mathbf{m}(k) - \widetilde{\mathbf{m}}(k)|. \tag{6.72}$$

This is summable because both $\mathbf{m}$ ad $\widetilde{\mathbf{m}}$ belong to $\mathbf{X}$ and since $\lim_{u\to\infty} b_{[t-u,t]}(i,k) = 0$, then $\mathbf{m} = \widetilde{\mathbf{m}}$. □

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